

# Yetter-Drinfeld category for the quasi-Turaev group coalgebra

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**Abstract.** Let  $\pi$  be a group. The aim of this paper is to construct the category of Yetter-Drinfeld modules over the quasi-Turaev group coalgebra  $H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, S, \Phi)$ , and prove that this category is isomorphic to the center of the representation category of  $H$ . Therefore a new Turaev braided group category is constructed.

**Keywords:** Yetter-Drinfeld module; Quasi-Hopf group coalgebra; Turaev braided group category; Center construction.

**Mathematics Subject Classification:** 16W30.

## Introduction

Given a group  $\pi$ , Turaev in [6] introduced the notion of a braided  $\pi$ -monoidal category which is called *Turaev braided group category* in this paper, and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory. Meanwhile such a category plays a key role in the construction of Hennings-type invariants of flat group-bundles over complements of link in the 3-sphere, see [8].

For the above reasons, it becomes very important to construct Turaev braided group category. Based on the work of [4], more results have been obtained in [2] and [9], where the method used in [4] was applied to weak Hopf algebras and regular multiplier Hopf algebras. It is well-known that there is another approach to the construction, for instance, in [1] the authors introduced the notion of quasi-Hopf group coalgebras and proved that the representation category of quasitriangular quasi-Hopf group coalgebras is exactly a Turaev braided group category.

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M. Zunino in [11] constructed the Yetter-Drinfeld category of crossed Hopf group coalgebra and showed that it is a Turaev braided group category. Motivated by this construction, in this paper, we will generalize this result to quasi-Turaev Hopf group coalgebra defined in [1]. The notion of Yetter-Drinfeld category of quasi-Turaev Hopf group coalgebra will be given, and the isomorphism between Yetter-Drinfeld category and the category of the center of representation category of quasi-Turaev Hopf group coalgebra will be established. Moreover, both of the categories are Turaev braided group categories.

This paper is organized as follows: In section 1, we will recall the notions of crossed  $T$ -category and its center and quasi-Turaev group coalgebra. In section 2, we will construct the Yetter-Drinfeld module over the quasi-Turaev group coalgebra and prove that the Yetter-Drinfeld category is isomorphic to the center of the representation category.

Throughout this article, let  $k$  be a fixed field. All the algebras and linear spaces are over  $k$ ; unadorned  $\otimes$  means  $\otimes_k$ .

## 1 Preliminary

In this section, we will recall the definitions and notations relevant to Turaev braided group categories.

### 1.1 Crossed $T$ -category

A tensor category  $\mathcal{C} = (C, \otimes, a, l, r)$  is a category  $\mathcal{C}$  endowed with a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (the tensor product), an object  $\mathcal{I} \in \mathcal{C}$  (the tensor unit), and natural isomorphisms  $a = a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$  for all  $U, V, W \in \mathcal{C}$  (the associativity constraint),  $l = l_U : \mathcal{I} \otimes U \rightarrow U$  (the left unit constraint) and  $r = r_U : U \otimes \mathcal{I} \rightarrow U$  (the right unit constraint) for all  $U \in \mathcal{C}$  such that for all  $U, V, W, X \in \mathcal{C}$ , the associativity pentagon

$$a_{U,V,W \otimes X} \circ a_{U \otimes V, W, X} = (U \otimes a_{V,W,X}) \circ a_{U,V \otimes W, X} \circ (a_{U,V,W} \otimes X),$$

and the triangle

$$(U \otimes l_V) \circ (r_U \otimes V) = a_{U, \mathcal{I}, V},$$

are satisfied. A tensor category  $\mathcal{C}$  is strict when all the constraints are identities.

Let  $\pi$  be a group with the unit 1. Recall from [1] that a crossed category  $\mathcal{C}$  (over  $\pi$ ) is given by the following data:

- $\mathcal{C}$  is a tensor category.
- A family of subcategory  $\{\mathcal{C}_\alpha\}_{\alpha \in \pi}$  such that  $\mathcal{C}$  is a disjoint union of this family and that  $U \otimes V \in \mathcal{C}_{\alpha\beta}$  for any  $\alpha, \beta \in \pi$ ,  $U \in \mathcal{C}_\alpha$  and  $V \in \mathcal{C}_\beta$ .
- A group homomorphism  $\varphi : \pi \rightarrow \text{aut}(\mathcal{C})$ ,  $\beta \mapsto \varphi_\beta$ , the *conjugation*, where  $\text{aut}(\mathcal{C})$  is the group of the invertible strict tensor functors from  $\mathcal{C}$  to itself, such that  $\varphi_\beta(\mathcal{C}_\alpha) = \mathcal{C}_{\beta\alpha\beta^{-1}}$  for any  $\alpha, \beta \in \pi$ .

We will use the left index notation in Turaev: Given  $\beta \in \pi$  and an object  $V \in \mathcal{C}_\alpha$ , the functor  $\varphi_\beta$  will be denoted by  $^\beta(\cdot)$  or  $^V(\cdot)$  and  $^{\beta^{-1}}(\cdot)$  will be denoted by  $^\overline{V}(\cdot)$ . Since  $^V(\cdot)$  is a functor, for any object  $U \in \mathcal{C}$  and any composition of morphism  $g \circ f$  in  $\mathcal{C}$ , we obtain  $^V id_U = id_{^V U}$  and  $^V(g \circ f) = ^V g \circ ^V f$ . Since the conjugation  $\varphi : \pi \rightarrow aut(\mathcal{C})$  is a group homomorphism, for any  $V, W \in \mathcal{C}$ , we have  $^{V \otimes W}(\cdot) = ^V(^W(\cdot))$  and  $^1(\cdot) = ^V(^{\overline{V}}(\cdot)) = ^{\overline{V}}(^V(\cdot)) = id_{\mathcal{C}}$ . Since for any  $V \in \mathcal{C}$ , the functor  $^V(\cdot)$  is strict, we have  $^V(f \otimes g) = ^V f \otimes ^V g$  for any morphism  $f$  and  $g$  in  $\mathcal{C}$ , and  $^V(1) = 1$ .

A *Turaev braided  $\pi$ -category* is a crossed  $T$ -category  $\mathcal{C}$  endowed with a braiding, i.e., a family of isomorphisms

$$c = \{c_{U,V} : U \otimes V \rightarrow ^V U \otimes V\}_{U,V \in \mathcal{C}}$$

obeying the following conditions:

- For any morphism  $f \in Hom_{\mathcal{C}_\alpha}(U, U')$  and  $g \in Hom_{\mathcal{C}_\beta}(V, V')$ , we have

$$(^{\alpha}g \otimes f) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g),$$

- For all  $U, V, W \in \mathcal{C}$ , we have

$$c_{U,V \otimes W} = a_{U,V,U,W}^{-1} \circ (^U V \otimes c_{U,W}) \circ a_{U,V,U,W} \circ (c_{U,V} \otimes W) \circ a_{U,V,W}^{-1}, \quad (1.1)$$

$$c_{U \otimes V,W} = a_{U \otimes V,W,U,V} \circ (c_{U,V,W} \otimes V) \circ a_{U,V,W,V}^{-1} \circ (U \otimes c'_{V,W}) \circ a_{U,V,W}. \quad (1.2)$$

- For any  $U, V \in \mathcal{C}$  and  $\alpha \in \pi$ ,  $\varphi_\alpha(c_{U,V}) = c_{\alpha U, \alpha V}$ .

## 1.2 The center of a crossed $T$ -category

Let  $\mathcal{C}$  be a crossed  $T$ -category. The center of  $\mathcal{C}$  is the braided crossed  $T$ -category  $\mathcal{Z}(\mathcal{C})$  defined as follows:

1. The objects of  $\mathcal{Z}(\mathcal{C})$  are the pairs  $(U, c_{U,-})$  satisfying the following conditions:
  - $U$  is an object of  $\mathcal{C}$ .
  - $c_{U,-}$  is a natural isomorphism from the functor  $U \otimes -$  to the functor  $^U(-) \otimes U$  such that for any objects  $V, W \in \mathcal{C}$ , the identity (1.1) is satisfied.
2. The morphism in  $\mathcal{Z}(\mathcal{C})$  from  $(U, c_{U,-})$  to  $(V, c'_{V,-})$  is a morphism  $f : U \rightarrow V$  such that for any object  $X \in \mathcal{C}$ ,

$$(^U X \otimes f) \circ c_{U,X} = c'_{V,X} \circ (f \otimes X). \quad (1.3)$$

The composition of two morphisms in  $\mathcal{Z}(\mathcal{C})$  is given by the composition in  $\mathcal{C}$ .

3. Given  $Z_1 = (U, c_{U,-})$  and  $Z_2 = (V, c'_{V,-})$  in  $\mathcal{Z}(\mathcal{C})$ , the tensor product  $Z_1 \otimes Z_2$  in  $\mathcal{Z}(\mathcal{C})$  is the couple  $(U \otimes V, (c \otimes c')_{U \otimes V, -})$ , where for any object  $W \in \mathcal{C}$ ,  $(c \otimes c')_{U \otimes V, -}$  is obtained by

$$(c \otimes c')_{U \otimes V, W} = a_{U \otimes V, W, U, V} \circ (c_{U, V} \otimes V) \circ a_{U, V, W, V}^{-1} \circ (U \otimes c'_{V, W}) \circ a_{U, V, W}. \quad (1.4)$$

4. The unit of  $\mathcal{Z}(\mathcal{C})$  is the couple  $(I, id_-)$ , where  $I$  is the unit of  $\mathcal{C}$ .
5. For any  $\alpha \in \pi$ , the  $\alpha$ th component of  $\mathcal{Z}(\mathcal{C})$ , denoted  $\mathcal{Z}_\alpha(\mathcal{C})$ , is the full subcategory of  $\mathcal{Z}(\mathcal{C})$  whose objects are the pairs  $(U, c_{U,-})$ , where  $U \in \mathcal{C}_\alpha$ .
6. For any  $\beta \in \pi$ , the automorphism  $\varphi_{\mathcal{Z}, \beta}$  is given by, for any  $(U, c_{U,-}) \in \mathcal{Z}(\mathcal{C})$ ,

$$\varphi_{\mathcal{Z}, \beta}(U, c_{U,-}) = (\varphi_\beta(U), \varphi_{\mathcal{Z}, \beta}(c_{U,-})), \quad (1.5)$$

where  $\varphi_{\mathcal{Z}, \beta}(c_{U,-})_{\varphi_\beta(U), X} = \varphi_\beta(c_{U, \varphi_\beta^{-1}(X)})$  for any  $X \in \mathcal{C}$ .

7. The braiding  $c$  in  $\mathcal{Z}(\mathcal{C})$  is obtained by setting  $c_{Z_1, Z_2} = c_{U, V}$  for any  $Z_1 = (U, c_{U,-})$ ,  $Z_2 = (V, c'_{V,-}) \in \mathcal{Z}(\mathcal{C})$ .

### 1.3 Quasi-Turaev group coalgebras

Recall from [1], a family of algebras  $H = \{H_\alpha\}_{\alpha \in \pi}$  is a quasi-semi-T-coalgebra if there exist a family of morphisms of algebra  $\Delta = \{\Delta_{\alpha, \beta} : H_{\alpha\beta} \rightarrow H_\alpha \otimes H_\beta\}_{\alpha, \beta \in \pi}$ , a morphism of algebra  $\varepsilon : H_1 \rightarrow k$  and a family of invertible elements  $\{\Phi_{\alpha, \beta, \gamma} \in H_\alpha \otimes H_\beta \otimes H_\gamma\}_{\alpha, \beta, \gamma \in \pi}$  such that

$$(H_\alpha \otimes \Delta_{\beta, \gamma})\Delta_{\alpha, \beta\gamma}(h)\Phi_{\alpha, \beta, \gamma} = \Phi_{\alpha, \beta, \gamma}(\Delta_{\alpha, \beta} \otimes H_\gamma)\Delta_{\alpha\beta, \gamma}(h), \quad (1.6)$$

$$(H_\alpha \otimes \varepsilon)(\Delta_{\alpha, 1}(a)) = a, \quad (\varepsilon \otimes H_\alpha)(\Delta_{1, \alpha}(a)) = a, \quad (1.7)$$

$$\begin{aligned} (1_\alpha \otimes \Phi_{\beta, \gamma, \lambda})(H_\alpha \otimes \Delta_{\beta, \gamma} \otimes H_\lambda)(\Phi_{\alpha, \beta\gamma, \lambda})(\Phi_{\alpha, \beta, \gamma} \otimes 1_\lambda) \\ = (H_\alpha \otimes H_\beta \otimes \Delta_{\gamma, \lambda})(\Phi_{\alpha, \beta, \gamma\lambda})(\Delta_{\alpha, \beta} \otimes H_\gamma \otimes H_\lambda)(\Phi_{\alpha\beta, \gamma, \lambda}), \end{aligned} \quad (1.8)$$

$$(H_\alpha \otimes \varepsilon \otimes H_\beta)(1_\alpha \otimes 1_1 \otimes 1_\beta) = 1_\alpha \otimes 1_\beta \quad (1.9)$$

for all  $h \in H_{\alpha\beta\gamma}$  and  $a \in H_\alpha$ .  $\Delta$  is called *comultiplication*, and  $\varepsilon$  the *counit*.

In our computations, we will use the Sweedler-Heyneman notation  $\Delta_{\alpha, \beta}(b) = b_{(1, \alpha)} \otimes b_{(2, \beta)}$  for all  $b \in H_{\alpha\beta}$  (summation implicitly understood). Since  $\Delta$  is only quasi-coassociative, we adopt further convention

$$(id_\alpha \otimes \Delta_{\beta, \gamma})\Delta_{\alpha, \beta\gamma}(h) = h_{(1, \alpha)} \otimes h_{(2, \beta\gamma)(1, \beta)} \otimes h_{(2, \beta\gamma)(2, \gamma)},$$

$$(\Delta_{\alpha, \beta} \otimes id_\gamma)\Delta_{\alpha\beta, \gamma}(h) = h_{(1, \alpha\beta)(1, \alpha)} \otimes h_{(1, \alpha\beta)(2, \beta)} \otimes h_{(2, \gamma)},$$

for all  $h \in H_{\alpha\beta\gamma}$ . We will denote the components of  $\Phi$  by capital letters, and the ones of  $\Phi^{-1}$  by small letters, namely,

$$\Phi_{\alpha,\beta,\gamma} = Y_\alpha^1 \otimes Y_\beta^2 \otimes Y_\gamma^3 = T_\alpha^1 \otimes T_\beta^2 \otimes T_\gamma^3 = \dots$$

$$\Phi_{\alpha,\beta,\gamma}^{-1} = y_\alpha^1 \otimes y_\beta^2 \otimes y_\gamma^3 = t_\alpha^1 \otimes t_\beta^2 \otimes t_\gamma^3 = \dots$$

A quasi-Hopf group coalgebra is a quasi-semi-T-coalgebra  $H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon)$  endowed with a family of invertible anti-automorphisms of algebra  $S = \{S_\alpha : H_\alpha \rightarrow H_{\alpha^{-1}}\}_{\alpha \in \pi}$  (*the antipode*) and elements  $\{p_\alpha, q_\alpha \in H_\alpha\}_{\alpha \in \pi}$  such that the following conditions hold:

$$S_\alpha(h_{(1,\alpha)})p_{\alpha^{-1}}h_{(2,\alpha^{-1})} = \varepsilon(h)p_{\alpha^{-1}}, \quad h_{(1,\alpha)}q_\alpha S_{\alpha^{-1}}(h_{(2,\alpha^{-1})}) = \varepsilon(h)q_\alpha, \quad (1.10)$$

$$Y_\alpha^1 q_\alpha S_{\alpha^{-1}}(Y_{\alpha^{-1}}^2) p_\alpha Y_\alpha^3 = 1_\alpha, \quad S_{\alpha^{-1}}(y_{\alpha^{-1}}^1) p_\alpha y_\alpha^2 q_\alpha S_{\alpha^{-1}}(y_{\alpha^{-1}}^3) = 1_\alpha. \quad (1.11)$$

A quasi-Turaev  $\pi$ -coalgebra is a quasi-Hopf  $\pi$ -coalgebra  $H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, \Phi)$  with a family of  $k$ -linear maps  $\varphi = \{\varphi_\beta : H_\alpha \rightarrow H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta \in \pi}$  (the crossing) such that the following conditions hold:

- For any  $\beta \in \pi$ ,  $\varphi_\beta$  is an algebra isomorphism.
- $\varphi_\beta$  preserves the comultiplication and the counit, i.e., for any  $\alpha, \beta, \gamma \in \pi$ ,

$$(\varphi_\beta \otimes \varphi_\beta) \Delta_{\alpha,\gamma} = \Delta_{\beta\alpha\beta^{-1}, \beta\gamma\beta^{-1}} \circ \varphi_\beta,$$

$$\varepsilon \circ \varphi_\beta = \varepsilon.$$

- $\varphi$  is multiplicative in the sense that  $\varphi_\beta \varphi_{\beta'} = \varphi_{\beta\beta'}$  for all  $\beta, \beta' \in \pi$ .
- The family  $\Phi$  is invariant under the crossing, i.e., for any  $\Phi_{\alpha,\beta,\gamma}$ ,

$$(\varphi_\eta \otimes \varphi_\theta \otimes \varphi_\vartheta) \Phi_{\alpha,\beta,\gamma} = \Phi_{\eta\alpha\eta^{-1}, \theta\beta\theta^{-1}, \vartheta\gamma\vartheta^{-1}}.$$

## 2 Main results

In this section, we will give the main result of this paper. First of all, we need some preparations. For any Hopf group coalgebra  $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$ , we obviously have the following identity

$$h_{(1,\alpha)} \otimes h_{(2,\beta)} S_{\beta^{-1}}(h_{(3,\beta^{-1})}) = h \otimes 1_\beta,$$

for all  $\alpha, \beta \in \pi$  and  $h \in H_\alpha$ . We will need the generalization of this formula to the quasi-Hopf group coalgebra setting. The following lemma will be given without proof.

**Lemma 2.1.** *Let  $H = (\{H_\alpha\}, \Delta, \varepsilon, S)$  be a quasi-Hopf group coalgebra. Set*

$$I_{\alpha,\beta}^R = I_\alpha^1 \otimes I_\beta^2 = y_\alpha^1 \otimes y_\beta^2 q_\beta S_{\beta^{-1}}(y_{\beta^{-1}}^3), \quad (2.1)$$

$$J_{\alpha,\beta}^R = J_\alpha^1 \otimes J_\beta^2 = Y_\alpha^1 \otimes S_\beta^{-1}(p_{\beta^{-1}} Y_{\beta^{-1}}^3) Y_\beta^2, \quad (2.2)$$

$$I_{\alpha,\beta}^L = \tilde{I}_\alpha^1 \otimes \tilde{I}_\beta^2 = Y_\alpha^2 S_\alpha^{-1}(Y_{\alpha^{-1}}^1 q_{\alpha^{-1}}) \otimes Y_\beta^3, \quad (2.3)$$

$$J_{\alpha,\beta}^L = \tilde{J}_\alpha^1 \otimes \tilde{J}_\beta^2 = S_{\alpha^{-1}}(y_{\alpha^{-1}}^1) p_\alpha y_\alpha^2 \otimes y_\beta^3. \quad (2.4)$$

*Then for all  $h \in H_\alpha$  and  $a \in H_\beta$ , we have*

$$\Delta_{\alpha,\beta}(h_{(1,\alpha\beta)}) I_{\alpha,\beta}^R [1 \otimes S_{\beta^{-1}}(h_{(2,\beta^{-1})})] = I_{\alpha,\beta}^R [h \otimes 1], \quad (2.5)$$

$$[1 \otimes S_\beta^{-1}(h_{(2,\beta^{-1})})] J_{\alpha,\beta}^R \Delta_{\alpha,\beta}(h_{(1,\alpha\beta)}) = [h \otimes 1] J_{\alpha,\beta}^R, \quad (2.6)$$

$$\Delta_{\alpha,\beta}(a_{(2,\alpha\beta)}) I_{\alpha,\beta}^L [S_\alpha^{-1}(a_{(1,\alpha^{-1})}) \otimes 1] = I_{\alpha,\beta}^L [1 \otimes a], \quad (2.7)$$

$$[S_{\alpha^{-1}}(a_{(1,\alpha^{-1})}) \otimes 1] J_{\alpha,\beta}^L \Delta_{\alpha,\beta}(a_{(2,\alpha\beta)}) = J_{\alpha,\beta}^L [1 \otimes a]. \quad (2.8)$$

*And the following relations hold:*

$$\Delta_{\alpha,\beta}(J_{\alpha\beta}^1) I_{\alpha,\beta}^R [1_\alpha \otimes S_{\beta^{-1}}(J_{\beta^{-1}}^2)] = 1_\alpha \otimes 1_\beta, \quad (2.9)$$

$$[1_\alpha \otimes S_\beta^{-1}(I_{\beta^{-1}}^2)] J_{\alpha,\beta}^R \Delta_{\alpha,\beta}(I_{\alpha\beta}^1) = 1_\alpha \otimes 1_\beta, \quad (2.10)$$

$$\Delta_{\alpha,\beta}(\tilde{J}_{\alpha\beta}^2) I_{\alpha,\beta}^L [S_\alpha^{-1}(\tilde{J}_{\alpha^{-1}}^1) \otimes 1_\beta] = 1_\alpha \otimes 1_\beta, \quad (2.11)$$

$$[S_{\alpha^{-1}}(\tilde{I}_{\alpha^{-1}}^1) \otimes 1_\beta] J_{\alpha,\beta}^L \Delta_{\alpha,\beta}(\tilde{I}_{\alpha\beta}^2) = 1_\alpha \otimes 1_\beta. \quad (2.12)$$

In [11], M. Zunino defined the Yetter-Drinfeld module over the crossed group coalgebra, and S. Majid in [3] ingeniously constructed the Yetter-Drinfeld module over quasi-Hopf algebra. With these help, we have the following definition.

**Definition 2.2.** *Fix an element  $\alpha \in \pi$ . An  $\alpha$ -Yetter-Drinfeld module or  $YD_\alpha$ -module is a couple  $V = \{V, \rho_V = \{\rho_{V,\lambda}\}_{\lambda \in \pi}\}$ , where  $\rho_{V,\lambda} : V \rightarrow V \otimes H_\lambda$ ,  $v \mapsto v_{(0,0)} \otimes v_{(1,\lambda)}$  is a  $k$ -linear morphism such that the following conditions are satisfied:*

1.  $V$  is a left  $H_\alpha$ -module,

2.  $V$  is counitary in the sense that

$$(id \otimes \varepsilon) \circ \rho_{V,1} = id. \quad (2.13)$$

3. For all  $v \in V$ ,

$$\begin{aligned} & (y_\alpha^2 \cdot v_{(0,0)})_{(0,0)} \otimes (y_\alpha^2 \cdot v_{(0,0)})_{(1,\lambda_1)} y_{\lambda_1}^1 \otimes y_{\lambda_2}^3 v_{(1,\lambda_2)} \\ &= \Phi_{\alpha,\lambda_1,\lambda_2}^{-1} \cdot [(y_\alpha^3 \cdot v)_{(0,0)} \otimes (y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(1,\lambda_1)} y_{\lambda_1}^1 \otimes (y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(2,\lambda_2)} y_{\lambda_2}^2]. \end{aligned} \quad (2.14)$$

4. For all  $h \in H_{\alpha\beta}$  and  $v \in V$ ,

$$h_{(1,\alpha)} \cdot v_{(0,0)} \otimes h_{(2,\beta)} v_{(1,\beta)} = (h_{(2,\alpha)} \cdot v)_{(0,0)} \otimes (h_{(2,\alpha)} \cdot v)_{(1,\beta)} \varphi_{\alpha^{-1}}(h_{(1,\alpha\beta\alpha^{-1})}). \quad (2.15)$$

**Remark 2.3.** Note that in the above definition, when the quasi-Hopf group coalgebra is trivial, i.e.,  $\varphi_{\alpha,\beta,\lambda} = 1_\alpha \otimes 1_\beta \otimes 1_\lambda$  for any  $\alpha, \beta, \lambda \in \pi$ , then we have a  $YD_\alpha$ -module over Hopf group coalgebra introduced in [11].

Given two  $YD_\alpha$ -modules  $(U, \rho_U)$  and  $(V, \rho_V)$ , a linear map  $f : U \rightarrow V$  is said to be a morphism of  $YD_\alpha$ -module if  $f$  is  $H_\alpha$ -linear and for any  $\lambda \in \pi$ ,

$$\rho_{V,\lambda} \circ f = (f \otimes H_\lambda) \circ \rho_{U,\lambda}.$$

Let  $YD(H)$  be the disjoint union of the categories  $YD_\alpha(H)$  for all  $\alpha \in \pi$ . The category  $YD(H)$  admits the structure of a braided  $T$ -category as follows:

- The tensor product of a  $YD_\alpha$ -module  $(V, \rho_V)$  and a  $YD_\beta$ -module  $(W, \rho_W)$  is a  $YD_{\alpha\beta}$ -module  $(V \otimes W, \rho_{V \otimes W})$ , where for any  $v \in V, w \in W$  and  $\lambda \in \pi$ ,

$$\begin{aligned} \rho_{V \otimes W}(v \otimes w) &= t_\alpha^1 Y_\alpha^1 \cdot (y_\alpha^2 \cdot v)_{(0,0)} \otimes t_\beta^2 \cdot (Y_\beta^3 y_\beta^3 \cdot w)_{(0,0)} \\ &\quad \otimes t_\lambda^3 (Y_\beta^3 y_\beta^3 \cdot w)_{(1,\lambda)} Y_\lambda^2 \varphi_{\beta^{-1}}((y_\alpha^2 \cdot v)_{(1,\beta\lambda\beta^{-1})}) y_\lambda^1. \end{aligned} \quad (2.16)$$

The unit of  $YD(H)$  is the pair  $(k, \rho_k)$ , where for any  $\lambda \in \pi$ ,  $\rho_\lambda(1) = 1 \otimes 1_\lambda$ . Then the tensor product of arrows is given by the tensor product of  $k$ -linear maps.

- For any  $\beta \in \pi$ , the conjugation functor  ${}^\beta(\cdot)$  is given as follows. Let  $(V, \rho_V)$  be a  $YD_\alpha$ -module and we set  ${}^\beta(V, \rho_V) = ({}^\beta V, \rho_{{}^\beta V})$ , where for any  $\lambda \in \pi$  and  $v \in V$ ,

$$\rho_{{}^\beta V}(v) = {}^\beta(({}^{\beta^{-1}}v)_{(0,0)}) \otimes \varphi_\beta(({}^{\beta^{-1}}v)_{(1,\beta^{-1}\lambda\beta)}). \quad (2.17)$$

For any morphism  $f : (V, \rho_V) \rightarrow (W, \rho_W)$  of  $YD$ -module and any  $v \in V$ , we set  $({}^\beta f)({}^\beta v) = {}^\beta(f(v))$ .

- For any  $YD_\alpha$ -module  $(V, \rho_V)$  and any  $YD_\beta$ -module  $(W, \rho_W)$ , the braiding  $c$  is given by

$$c_{V,W}(v \otimes w) = {}^\alpha[J_{(1,\beta)}^1 y_\beta^1 S_{\beta^{-1}}(J_{\beta^{-1}}^2 y_{\beta^{-1}}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\beta^{-1})} \tilde{I}_{\beta^{-1}}^1 \cdot w) \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)}]. \quad (2.18)$$

**Lemma 2.4.** For a fixed element  $\alpha \in \pi$ , let  $(V, c_{V,-})$  be any object in  $\mathcal{Z}_\alpha(\text{Rep}(H))$ . For any  $\lambda \in \pi$ , define the linear map  $\rho_{V,\lambda} : V \rightarrow V \otimes H_\lambda$  by

$$\rho_{V,\lambda}(v) = c_{V,H_\lambda}^{-1}({}^\alpha 1_\lambda \otimes v). \quad (2.19)$$

Then the pair  $V = (V, \rho_V = \{\rho_{V,\lambda}\}_{\lambda \in \pi})$  is a  $YD_\alpha$ -module. Hence we have a functor  $F_1 : \mathcal{Z}(\text{Rep}(H)) \rightarrow YD(H)$  given by  $F_1(V, c_{V,-}) = (V, \rho_V)$  and  $F_1(f) = f$ , where  $f$  is a morphism in  $\mathcal{Z}(\text{Rep}(H))$ .

*Proof.* We just need to verify that  $(V, \rho_V)$  satisfies the axioms of  $YD_\alpha$ -modules.

First of all, for any  $\lambda_1, \lambda_2 \in \pi$ , consider  $H_{\lambda_1}$  and  $H_{\lambda_2}$  as the modules over themselves. By (1.1), we have

$$a_{V,H_{\lambda_1},H_{\lambda_2}}^{-1} \circ c_{V,H_{\lambda_1} \otimes H_{\lambda_2}}^{-1} \circ a_{V,H_{\lambda_1},{}^V H_{\lambda_2},V}^{-1} = (c_{V,H_{\lambda_1}}^{-1} \otimes H_{\lambda_2}) \circ a_{V,H_{\lambda_1},V,H_{\lambda_2}}^{-1} \circ ({}^V H_{\lambda_1} \otimes c_{V,H_{\lambda_2}}^{-1}).$$

For all  $v \in V$ , both of the sides evaluating at  ${}^\alpha 1_{\lambda_1} \otimes {}^\alpha 1_{\lambda_2} \otimes v$ , we have

$$\begin{aligned} & (y_\alpha^2 \cdot v_{(0,0)})_{(0,0)} \otimes (y_\alpha^2 \cdot v_{(0,0)})_{(1,\lambda_1)} y_{\lambda_1}^1 \otimes y_{\lambda_2}^3 v_{(1,\lambda_2)} \\ &= y_{\alpha,\lambda_1,\lambda_2} \cdot [(y_\alpha^3 \cdot v)_{(0,0)} \otimes (y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(1,\lambda_1)} y_{\lambda_1}^1 \otimes (y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(2,\lambda_2)} y_{\lambda_2}^2]. \end{aligned}$$

The counitarity of  $V$  is obvious.

Secondly for all  $v \in V$  and  $h \in H_{\alpha\lambda}$ , we have on one hand,

$$h \cdot c_{V,H_\lambda}^{-1}({}^\alpha 1_\lambda \otimes v) = h \cdot (v_{(0,0)} \otimes v_{(1,\lambda)}) = h_{(1,\alpha)} \cdot v_{(0,0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)},$$

and on the other hand,

$$\begin{aligned} c_{V,H_\lambda}^{-1}(h \cdot ({}^\alpha 1_\lambda \otimes v)) &= c_{V,H_\lambda}^{-1}(h_{(1,\alpha\lambda\alpha^{-1})} \cdot {}^\alpha 1_\lambda \otimes h_{(2,\alpha)} \cdot v) \\ &= c_{V,H_\lambda}^{-1}({}^\alpha(\varphi_{\alpha^{-1}}(h_{(1,\alpha\lambda\alpha^{-1})})) \otimes h_{(2,\alpha)} \cdot v) \\ &= (h_{(2,\alpha)} \cdot v)_{(0,0)} \otimes (h_{(2,\alpha)} \cdot v)_{(1,\lambda)} \varphi_{\alpha^{-1}}(h_{(1,\alpha\lambda\alpha^{-1})}). \end{aligned}$$

Since the braiding  $c_{V,H_\lambda}$  is  $H$ -linear, we obtain

$$h_{(1,\alpha)} \cdot v_{(0,0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)} = (h_{(2,\alpha)} \cdot v)_{(0,0)} \otimes (h_{(2,\alpha)} \cdot v)_{(1,\lambda)} \varphi_{\alpha^{-1}}(h_{(1,\alpha\lambda\alpha^{-1})}).$$

Finally, let  $f : (V, c_{V,-}) \rightarrow (W, c_{W,-})$  is a morphism in  $\mathcal{Z}_\alpha(\text{Rep}(H))$ , then as the case of Hopf group coalgebra,  $f$  gives rise to a morphism of  $\text{YD}_\alpha$ -module. It is easy to see that  $F_1$  is a functor. This completes the proof.  $\square$

Assume that  $(V, \rho_V)$  is an object in the category  $\text{YD}_\alpha(H)$ . For any  $\lambda \in \pi$  and left  $H_\lambda$ -module  $X$ , give the linear map  $c_{V,X} : V \otimes X \rightarrow {}^\alpha X \otimes V$  by

$$c_{V,X}(v \otimes x) = {}^\alpha[J_{(1,\lambda)}^1 y_\lambda^1 S_{\lambda^{-1}}(J_{\lambda^{-1}}^2 y_{\lambda^{-1}}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\lambda^{-1})} \tilde{I}_{\lambda^{-1}}^1) \cdot x] \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)},$$

for all  $v \in V$  and  $x \in X$ .

**Lemma 2.5.** *The couple  $(V, c_{V,-})$  is an object in  $\mathcal{Z}(\text{Rep}(H))$ . Hence we have a functor  $F_2 : \text{YD}(H) \rightarrow \mathcal{Z}(\text{Rep}(H))$  given by  $F_2(V, \rho_V) = (V, c_{V,-})$  and  $F_2(f) = f$ , where  $f$  is a morphism in  $\text{YD}(H)$ . The functors  $F_1$  and  $F_2$  are inverses.*

*Proof.* Firstly for any  $\lambda \in \pi$  and left  $H_\lambda$ -module  $X$ , we set a morphism  $\hat{c}_{V,X} : {}^\alpha X \otimes V \rightarrow V \otimes X$  by

$$\hat{c}_{V,X}({}^\alpha x \otimes v) = v_{(0,0)} \otimes v_{(1,\lambda)} \cdot x.$$



Then

$$\begin{aligned}
& \hat{c}_{V,X} \circ c_{V,X}(v \otimes x) \\
&= \hat{c}_{V,X}(\alpha[J_{(1,\lambda)}^1 y_\lambda^1 S_{\lambda-1}(J_{\lambda-1}^2 y_{\lambda-1}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\lambda-1)} \tilde{I}_{\lambda-1}^1) \cdot x] \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)}) \\
&= \underline{[J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)}]}_{(0,0)} \otimes \\
&\quad \underline{[J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)}]}_{(1,\lambda)} \varphi_{\alpha-1}(J_{(1,\alpha\lambda\alpha-1)}^1) \varphi_{\alpha-1}(y_{\alpha\lambda\alpha-1}^1) \cdot [S_{\lambda-1}(J_{\lambda-1}^2 y_{\lambda-1}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\lambda-1)} \tilde{I}_{\lambda-1}^1) \cdot x] \\
&\stackrel{(2.14)}{=} J_{(1,\alpha)}^1 \cdot \underline{(y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)})}_{(0,0)} \otimes \\
&\quad J_{(2,\lambda)}^1 \underline{(y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot v)_{(0,0)})}_{(1,\lambda)} y_\lambda^1 S_{\lambda-1}(J_{\lambda-1}^2 y_{\lambda-1}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\lambda-1)} \tilde{I}_{\lambda-1}^1) \cdot x \\
&= J_{(1,\alpha)}^1 t_\alpha^1 \cdot (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(0,0)} \otimes \\
&\quad J_{(2,\lambda)}^1 t_\lambda^2 (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(1,1)(1,\lambda)} y_\lambda^1 S_{\lambda-1}(J_{\lambda-1}^2 t_{\lambda-1}^{-3} (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(1,1)(2,\lambda-1)} y_{\lambda-1}^2) \tilde{I}_{\lambda-1}^1 \cdot x \\
&= J_{(1,\alpha)}^1 t_\alpha^1 \cdot (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(0,0)} \otimes \\
&\quad J_{(2,\lambda)}^1 t_\lambda^2 (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(1,1)(1,\lambda)} y_\lambda^1 S_{\lambda-1}(y_{\lambda-1}^2 \tilde{I}_{\lambda-1}^1) S_{\lambda-1}(J_{\lambda-1}^2 t_{\lambda-1}^{-3} (y_\alpha^3 \tilde{I}_\alpha^2 \cdot v)_{(1,1)(2,\lambda-1)}) \cdot x \\
&= J_{(1,\alpha)}^1 t_\alpha^1 \cdot v_{(0,0)} \otimes J_{(2,\lambda)}^1 t_\lambda^2 v_{(1,1)(1,\lambda)} q_\lambda S_{\lambda-1}(v_{(1,1)(2,\lambda-1)}) S_{\lambda-1}(J_{\lambda-1}^2 t_{\lambda-1}^{-3}) \cdot x \\
&= J_{(1,\alpha)}^1 t_\alpha^1 \cdot v \otimes J_{(2,\lambda)}^1 t_\lambda^2 q_\lambda S_{\lambda-1}(J_{\lambda-1}^2 t_{\lambda-1}^{-3}) \cdot x \\
&= v \otimes x.
\end{aligned}$$

Hence  $\hat{c}_{V,X} \circ c_{V,X} = id_{V \otimes X}$ . Similarly  $\hat{c}_{V,X} \circ c_{V,X} = id_{\alpha_X \otimes V}$ . Therefore  $\hat{c}_{V,X}$  and  $c_{V,X}$  are inverses.

Secondly for any  $h \in H_{\alpha\lambda}$ ,

$$\begin{aligned}
h \cdot c_{V,X}^{-1}(\alpha x \otimes v) &= h_{(1,\alpha)} \cdot v_{(0,0)} \otimes h_{(2,\lambda)} v_{(1,\lambda)} \cdot x \\
&\stackrel{(2.15)}{=} (h_{(2,\alpha)} \cdot v)_{(0,0)} \otimes (h_{(2,\alpha)} \cdot v)_{(1,\beta)} \varphi_{\alpha-1}(h_{(1,\alpha\lambda\alpha-1)}) \cdot x \\
&= c_{V,X}^{-1}(\alpha(\varphi_{\alpha-1}(h_{(1,\alpha\lambda\alpha-1)}) \cdot x) \otimes h_{(2,\alpha)} \cdot v) \\
&= c_{V,X}^{-1}(h_{(1,\alpha\lambda\alpha-1)} \cdot \alpha x \otimes h_{(2,\alpha)} \cdot v) \\
&= c_{V,X}^{-1}(h \cdot (\alpha x \otimes v)).
\end{aligned}$$

That is,  $c_{V,X}^{-1}$  is  $H_{\alpha\lambda}$ -linear, so is  $c_{V,X}$ . The naturality of  $c_{V,X}$  is straightforward to verify.

Next suppose that  $X_1$  is an  $H_{\lambda_1}$ -module and  $X_2$  an  $H_{\lambda_2}$ -module for all  $\lambda_1, \lambda_2 \in \pi$ , and for any  $x_1 \in X_1, x_2 \in X_2$ ,

$$\begin{aligned}
& a_{V,X_1,X_2} \circ (c_{V,X_1}^{-1} \otimes X_2) \circ a_{\alpha X_1,V,X_2}^{-1} \circ (\alpha X_1 \otimes c_{V,X_2}^{-1}) \circ a_{\alpha X_1,\alpha X_2,V}(\alpha x_1 \otimes^\alpha x_2 \otimes v) \\
&= T_\alpha^1 \cdot [y_\alpha^2 \cdot (Y_\alpha^3 \cdot v)_{(0,0)}]_{(0,0)} \otimes T_{\lambda_1}^2 \cdot [y_\alpha^2 \cdot (Y_\alpha^3 \cdot v)_{(0,0)}]_{(1,\lambda_1)} Y_{\lambda_1}^{-1} Y_{\lambda_1}^1 \cdot x_1 \\
&\quad \otimes T_{\lambda_2}^3 y_{\lambda_2}^3 (Y_\alpha^3 \cdot v)_{(1,\lambda_2)} Y_{\lambda_2}^2 \cdot x_2 \\
&\stackrel{(2.14)}{=} (y_\alpha^3 Y_\alpha^3 \cdot v)_{(0,0)} \otimes (y_\alpha^3 Y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(1,\lambda_1)} y_{\lambda_1}^1 Y_{\lambda_1}^1 \cdot x_1 \otimes (y_\alpha^3 Y_\alpha^3 \cdot v)_{(1,\lambda_1\lambda_2)(2,\lambda_2)} y_{\lambda_2}^2 Y_{\lambda_2}^2 \cdot x_2 \\
&= v_{(0,0)} \otimes v_{(1,\lambda_1\lambda_2)(1,\lambda_1)} \cdot x_1 \otimes v_{(1,\lambda_1\lambda_2)(2,\lambda_2)} y_{\lambda_2}^2 \cdot x_2 \\
&= c_{V,X_1 \otimes X_2}^{-1}(\alpha x_1 \otimes^\alpha x_2 \otimes v).
\end{aligned}$$

Let  $V, W$  be  $YD_\alpha$ -modules,  $f : V \rightarrow W$  be any morphism of  $YD_\alpha$ -module. For any  $H_\lambda$ -module  $X$  and  $x \in X$ ,

$$\begin{aligned}
c_{W,X} \circ (f \otimes id)(v \otimes x) &= c_{W,X}(f(v) \otimes x) \\
&= {}^\alpha [J_{(1,\lambda)}^1 y_\lambda^1 S_{\lambda^{-1}}(J_{\lambda^{-1}}^2 y_{\lambda^{-1}}^3 (\tilde{I}_\alpha^2 \cdot f(v))_{(1,\lambda^{-1})} \tilde{I}_{\lambda^{-1}}^1 \cdot x) \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot (\tilde{I}_\alpha^2 \cdot f(v))_{(0,0)}] \\
&= {}^\alpha [J_{(1,\lambda)}^1 y_\lambda^1 S_{\lambda^{-1}}(J_{\lambda^{-1}}^2 y_{\lambda^{-1}}^3 (f(\tilde{I}_\alpha^2 \cdot v))_{(1,\lambda^{-1})} \tilde{I}_{\lambda^{-1}}^1 \cdot x) \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot (f(\tilde{I}_\alpha^2 \cdot v))_{(0,0)}] \\
&= {}^\alpha [J_{(1,\lambda)}^1 y_\lambda^1 S_{\lambda^{-1}}(J_{\lambda^{-1}}^2 y_{\lambda^{-1}}^3 (\tilde{I}_\alpha^2 \cdot v)_{(1,\lambda^{-1})} \tilde{I}_{\lambda^{-1}}^1 \cdot x) \otimes J_{(2,\alpha)}^1 y_\alpha^2 \cdot f((\tilde{I}_\alpha^2 \cdot v)_{(0,0)})] \\
&= {}^\alpha (id \otimes f) c_{V,X}(v \otimes x).
\end{aligned}$$

That is,  $f$  is a morphism in  $\mathcal{Z}(\text{Rep}(H))$ . Finally by similar arguments in [11], we know that  $F_1$  and  $F_2$  are inverses. This completes the proof.  $\square$

**Theorem 2.6.** *The category  $YD(H)$  is isomorphic to the category  $\mathcal{Z}(\text{Rep}(H))$ . This isomorphism induces the structure of braided  $T$ -category on  $YD(H)$ .*

*Proof.* This isomorphism holds via the functors  $F_1$  and  $F_2$ .

Let  $(V, \rho_V)$  be a  $YD_\alpha$ -module and  $(W, \rho_W)$  be a  $YD_\beta$ -module. Suppose that  $(V, c_{V,-}) = F_2(V, \rho_V)$  and  $(W, c'_{W,-}) = F_2(W, \rho_W)$  and set

$$(V, \rho_V) \otimes (W, \rho_W) = F_1(F_2(V, \rho_V) \otimes F_2(W, \rho_W)) = F_1(V \otimes W, (c \otimes c')_{V \otimes W, -}).$$

For any  $v \in V, w \in W$ , we have

$$\begin{aligned}
\rho_{V \otimes W, \lambda}(v \otimes w) &= ((c \otimes c')_{V \otimes W, H_\lambda})^{-1}({}^{\alpha\beta} 1_\lambda \otimes v \otimes w) \\
&= a_{V, W, H_\lambda}^{-1} \circ (V \otimes c'_{W, H_\lambda}) \circ a_{V, \beta H_\lambda, W} \circ (c_{V, \beta H_\lambda}^{-1} \otimes W) \circ a_{\alpha\beta H_\lambda, V, W}^{-1}({}^{\alpha\beta} 1_\lambda \otimes v \otimes w) \\
&= y_\alpha^1 Y_\alpha^1 \cdot (t_\alpha^2 \cdot v)_{(0,0)} \otimes y_\beta^2 \cdot (Y_\beta^3 t_\beta^3 \cdot w)_{(0,0)} \\
&\quad \otimes y_\lambda^3 (Y_\beta^3 t_\beta^3 \cdot w)_{(1,\lambda)} Y_\lambda^2 \varphi_{\beta^{-1}}((t_\alpha^2 \cdot v)_{(1,\beta\lambda\beta^{-1})}) t_\lambda^1,
\end{aligned}$$

where

$$\begin{aligned}
(c_{V, \beta H_\lambda}^{-1} \otimes W)({}^{\alpha\beta} t_\lambda^1 \otimes t_\alpha^2 \cdot v \otimes t_\beta^3 \cdot w) \\
= (t_\alpha^2 \cdot v)_{(0,0)} \otimes {}^\beta [\varphi_{\beta^{-1}}(t_\alpha^2 \cdot v)_{(1,\beta\lambda\beta^{-1})} t_\lambda^1] \otimes t_\beta^3 \cdot w.
\end{aligned}$$

The part concerning the tensor unit of  $YD(H)$  is trivial. By similar arguments in [11], we can verify the condition (2.16)-(2.18). This completes the proof.  $\square$

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